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On the leading nonlinear correction to gravity-wave dynamics

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The principal nonlinear correction to the dynamics of gravity waves on an irrotational fluid is traditionally derived as a non-resonant perturbation solution to the Stokes expansion. When the problem is reformulated in the Hamiltonian description and limited to moderately collimated random waves over infinite depth, the perturbation term assumes a very simple and descriptive form. The sum-frequency component for the surface height is just a bilinear product of the height with the associated scalar strain, and the accompanying term in the potential is half the time derivative of the squared linear height. This solution is exact in one surface dimension and remains quite accurate for long-crested waves in two dimensions, with an error small to second order in the angular spread of constituent wave vectors. It is a natural generalization for random, disordered wave ensembles of the second-order Stokes solution, and its effect is to sharpen the random crests and to flatten the troughs. For wave sets of narrow relative bandwidth the difference-frequency component consists of a negligible elevation term and a non-negligible potential term whose gradient is the surface value of the volume return flow balancing the quadratic wave transport of fluid.

1. Introduction

Even the casual observer of waves on lakes or the ocean is aware that the crests are sharper than the troughs, and we take for granted that the nonlinearity of the dynamics is responsible for steepening the slopes near the crests through a local rearrangement of the flow. For an explanation that goes beyond the Stokes solution for a steady monochrome wave and extends to random wave sets, we might look to the leading non-resonant perturbation correction, which is driven by – and more-or-less passively accompanies – the linear solution (Phillips 1960). Unfortunately this leading correction, whose general form is easily derived and has long been available (Longuet-Higgins 1963; Webber & Barrick 1977), resists easy interpretation because of the detailed structure of its Fourier representation. The problem is that the general correction has to account for the mutual interaction of all possible pairs of waves in a heterogeneous set, not just the 'self-interaction' responsible for the shape of an individual waveform. For instance, the to-and-fro transport and amplitude modulation of short waves on the orbital currents of long waves is part of this solution, though these effects are better described by other approaches (Longuet-Higgins & Stewart 1960, 1962).

All the non-resonant behaviour, in which the quadratic terms are the leading order, can in principle be absorbed by a canonical transformation of the field equations; Krasitskii (1994) has used such a procedure to derive a series expansion in powers of the linear solution. However the quadratic term requires a sum over all pairs of wavenumbers, just as in the leading perturbation solution. Creamer *et al.* (1989) have devised an implicit integral form for the canonical transformation in one surface dimension that they show to be capable of displaying crest sharpening and orbital wave advection, but the arithmetic is subtle and evidently not formally extendable to two surface dimensions. Stokes expansions of nearly monochromatic wave trains with slowly varying amplitude and phase have been carried out in one surface dimension by Tayfun (1986) and Trulsen & Dysthe (1996); these reproduce the usual solution locally along with corrections proportional to the rate of variation.

When the problem is reformulated in the Hamiltonian description and limited to moderately collimated random waves over infinite depth, the perturbation solution assumes a simple and easily interpretable form, one that appears to have been unnoticed in the standard formulation. It consists of terms that are simple quadratic products of quantities from the linear solution; in one surface dimension this solution is exact for arbitrary random wave sets, so long as these contain progressive waves propagating in a common direction. Its form is a direct generalization of the narrow-band second-order Stokes wave. In two surface dimensions the solution is approximate, with residual errors that are small to second order in the angular spread of the wave sets, so that the solution remains good for long-crested random waves.

As with all products of linear fields, the solution is composed of sum-frequency and difference-frequency components – see for example Forristall (2000); these can be specified independently, and in fact a consistent perturbation solution requires that they be constructed individually. In the difference-frequency solution the velocity potential plays a prominent role, supplying the volume return flow balancing the quadratic wave transport of fluid. While this result is unsurprising, its formal demonstration at second order is believed to be new.

2. Formulation

The Hamiltonian description of deep-water irrotational wave motion in the form introduced by Watson & West (1975) and elaborated by Milder (1990) provides a pair of evolution equations for the surface elevation $\zeta(\boldsymbol{x}, t)$ and the surface value $\phi(\boldsymbol{x}, t)$ of the velocity potential, which through first nonlinear order are

$$\frac{\partial \zeta}{\partial t} - \hat{\mathrm{k}}\phi + \nabla\cdot(\zeta\nabla\phi) + \hat{\mathrm{k}}(\zeta\hat{\mathrm{k}}\phi) = 0, \qquad \frac{\partial \phi}{\partial t} + g\zeta + \frac{1}{2}[(\nabla\phi)^2 - (\hat{\mathrm{k}}\phi)^2] = 0; \quad (2.1)$$

$\hat{\mathrm{k}}$ is the linear operator that multiplies each spatial Fourier component by its wave-number modulus, ∇ is the two-dimensional horizontal gradient; and g is the acceleration due to gravity. These equations, equivalent to the usual kinematic and dynamical boundary conditions, are a canonical pair derivable from the Hamiltonian

$$H[\zeta, \phi] = \frac{1}{2}\int \{\phi\hat{\mathrm{k}}\phi + g\zeta^2 + \zeta[(\nabla\phi)^2 - (\hat{\mathrm{k}}\phi)^2]\}\,\mathrm{d}\boldsymbol{x}. \quad (2.2)$$

The horizontal and vertical orbital currents at the surface are, in this description

$$\boldsymbol{u} = \nabla\phi - w\nabla\zeta, \qquad w = (1 + [\zeta, \hat{\mathrm{k}}] + \cdots)\hat{\mathrm{k}}\phi, \quad (2.3)$$

where $[\zeta, \hat{\mathrm{k}}] = \zeta\hat{\mathrm{k}} - \hat{\mathrm{k}}\zeta$ is the commutator product, an operator in which $\hat{\mathrm{k}}$ is understood to be applied to the entire expression to its right. The density of wave momentum, or mass transport, is exactly

$$\boldsymbol{m} = -\phi\nabla\zeta. \quad (2.4)$$

Inserting

$$\zeta = \zeta_0 + \zeta_1 + \cdots, \qquad \phi = \phi_0 + \phi_1 + \cdots, \tag{2.5}$$

into (2.1), we obtain for the leading nonlinear terms

$$\frac{\partial \zeta_1}{\partial t} - \hat{\mathrm{k}}\phi_1 = f, \qquad \frac{\partial \phi_1}{\partial t} + g\zeta_1 = -p, \tag{2.6}$$

in which the quantities

$$f = -[\nabla \cdot (\zeta_0 \nabla \phi_0) + \hat{\mathrm{k}}(\zeta_0 \hat{\mathrm{k}}\phi_0)], \qquad p = \frac{1}{2}[(\nabla\phi_0)^2 - (\hat{\mathrm{k}}\phi_0)^2] \tag{2.7}$$

can be interpreted respectively as an excess fluid flux and pressure, both quadratic in the linear solution

$$\zeta_0 = \mathrm{Re}\psi, \quad \phi_0 = \mathrm{Re}(-\mathrm{i}\hat{\mathrm{c}}\psi), \tag{2.8}$$

where $\hat{\mathrm{c}} = \sqrt{g/\hat{\mathrm{k}}}$ is the phase-speed operator, and where

$$\psi = \int a(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\theta_k} \, \mathrm{d}\boldsymbol{k}_c, \quad \theta_{\boldsymbol{k}}(\boldsymbol{x}, t) = \boldsymbol{k} \cdot \boldsymbol{x} - \omega_k t \tag{2.9}$$

with

$$\omega_k = \sqrt{gk}, \quad \mathrm{d}\boldsymbol{k}_c = \mathrm{d}\boldsymbol{k}/4\pi^2. \tag{2.10}$$

3. Sum and difference beats

The product of any two quantities related to the linear solution (2.8) produces terms with sums and differences of wavenumber and frequency pairs (*sum* and *difference beats*). For example, the squared elevation

$$q = \zeta_0^2 = q^{(+)} + q^{(-)} \tag{3.1}$$

is, according to (2.9),

$$\left. \begin{aligned} q^{(+)} &= \frac{1}{2} \mathrm{Re}\, \psi^2 = \frac{1}{2} \mathrm{Re} \iint a a' \, \mathrm{e}^{\mathrm{i}(\theta + \theta')} \, \mathrm{d}\boldsymbol{k}_c \, \mathrm{d}\boldsymbol{k}_c', \\ q^{(-)} &= \frac{1}{2} \mathrm{Re}\, \psi^* \psi = \frac{1}{2} \mathrm{Re} \iint a^* a' \, \mathrm{e}^{\mathrm{i}(\theta' - \theta)} \, \mathrm{d}\boldsymbol{k}_c \, \mathrm{d}\boldsymbol{k}_c', \end{aligned} \right\} \tag{3.2}$$

and for the usual gravity-wave dispersion relation (2.10) these components are non-overlapping in the wavenumber–frequency domain. The source terms in the perturbation equation (2.6) consist of combinations of terms analogous to $q^{(\pm)}$ above, and the solutions can be seen to have the form

$$\zeta_1^{(\pm)} = \frac{1}{2} \mathrm{Re} \iint (a, a^*) a' Z^{(\pm)}(\boldsymbol{k}, \boldsymbol{k}') \, \mathrm{e}^{\mathrm{i}(\theta \pm \theta')} \, \mathrm{d}\boldsymbol{k}_c \, \mathrm{d}\boldsymbol{k}_c', \tag{3.3}$$

and so on. While it is straightforward to work out all the necessary double Fourier coefficients so as to recognize among them the approximate solution below, it will be simpler to present the solution and demonstrate directly that it satisfies the perturbation equations. In doing so we shall rely on the usual relations among the linear fields (2.8) and their time derivatives:

$$\frac{\partial \zeta_0}{\partial t} - \hat{\mathrm{k}}\phi_0 = 0, \qquad \frac{\partial \phi_0}{\partial t} + g\zeta_0 = 0. \tag{3.4}$$

4. The sum-beat solution

The following simple sum-frequency products of the linear fields will now be shown to satisfy the perturbation equation to a good approximation for long-crested random waves:

$$\left.\begin{aligned}\zeta_1^{(+)} &= (\zeta_0\hat{\mathrm{k}}\zeta_0)^{(+)} = \frac{1}{2}\,\mathrm{Re}\psi\hat{\mathrm{k}}\psi,\\ \phi_1^{(+)} &= (\zeta_0\hat{\mathrm{k}}\phi_0)^{(+)} = \frac{1}{2}\left(\frac{\partial\zeta_0^2}{\partial t}\right)^{(+)} = \frac{1}{4}\frac{\partial}{\partial t}\,\mathrm{Re}\psi^2.\end{aligned}\right\} \tag{4.1}$$

Using (3.4) and $\hat{\mathrm{k}}^2 = -\nabla^2$ to evaluate the time derivative

$$\begin{aligned}\frac{\partial\zeta_1}{\partial t} &= (\hat{\mathrm{k}}\phi_0)(\hat{\mathrm{k}}\zeta_0) + \zeta_0\hat{\mathrm{k}}^2\phi_0\\ &= (\hat{\mathrm{k}}\phi_0)(\hat{\mathrm{k}}\zeta_0) + (\nabla\phi_0)\cdot(\nabla\zeta_0) - \nabla\cdot\zeta_0\nabla\phi_0,\end{aligned} \tag{4.2}$$

and subtracting $\hat{\mathrm{k}}\phi_1 = \hat{\mathrm{k}}\zeta_0\hat{\mathrm{k}}\phi_0$, we have

$$\frac{\partial\zeta_1}{\partial t} - \hat{\mathrm{k}}\phi_1 = f + r_\zeta, \qquad r_\zeta = (\hat{\mathrm{k}}\phi_0)(\hat{\mathrm{k}}\zeta_0) + (\nabla\phi_0)\cdot(\nabla\zeta_0). \tag{4.3}$$

For the sum-frequency domain the residual error has the form

$$r_\zeta^{(+)} = \frac{1}{2}\,\mathrm{Re}\iint ab'(kk' - \boldsymbol{k}\cdot\boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}(\theta+\theta')}\,\mathrm{d}\boldsymbol{k}_c\,\mathrm{d}\boldsymbol{k}_c', \qquad b = -\mathrm{i}c_k a; \tag{4.4}$$

similarly

$$\frac{\partial\phi_1}{\partial t} = (\hat{\mathrm{k}}\phi_0)^2 - g\zeta_0\hat{\mathrm{k}}\zeta_0 \tag{4.5}$$

or

$$\frac{\partial\phi_1}{\partial t} + g\zeta_1 = (\hat{\mathrm{k}}\phi_0)^2 = -p + r_\phi, \quad r_\phi = \frac{1}{2}[(\hat{\mathrm{k}}\phi_0)^2 + (\nabla\phi_0)^2], \tag{4.6}$$

with a residual error

$$r_\phi^{(+)} = \frac{1}{4}\,\mathrm{Re}\iint bb'(kk' - \boldsymbol{k}\cdot\boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}(\theta+\theta')}\,\mathrm{d}\boldsymbol{k}_c\,\mathrm{d}\boldsymbol{k}_c'. \tag{4.7}$$

Note that in one surface dimension, when the wave vectors share a common direction, the expression $(kk' - \boldsymbol{k}\cdot\boldsymbol{k}')$ vanishes identically along with both residual terms, so that the solution (4.1) is exact. It can be interpreted as the random-wave generalization to the second-order Stokes solution, as will be discussed below. This result breaks down for shallow water, when the primary waves feel the bottom more strongly than the second-harmonic perturbations. In two surface dimensions the result survives as an accurate approximation when the waves are long-crested, that is, directionally collimated with small relative angles $\varphi_{\boldsymbol{kk}'}$; then the relative residual error in the source terms, as expressed by the ratio

$$\frac{(kk' - \boldsymbol{k}\cdot\boldsymbol{k}')}{(kk' + \boldsymbol{k}\cdot\boldsymbol{k}')} = \tan^2(\varphi_{\boldsymbol{k},\boldsymbol{k}'}/2), \tag{4.8}$$

is small to second order in $\varphi_{\boldsymbol{k},\boldsymbol{k}'}$.

5. The difference-beat solution

The linear relations (3.4) can be used to rewrite the quadratic flux and pressure (2.7) in the form

$$\left.\begin{aligned} f &= -\nabla\cdot\boldsymbol{m} - \nabla^2(\zeta_0\phi_0) - \frac{1}{2}\hat{\mathrm{k}}\left(\frac{\partial}{\partial t}\zeta_0^2\right),\\ p &= \frac{1}{4}\left[\nabla^2\phi_0^2 - \left(\frac{\partial}{\partial t}\right)^2\zeta_0^2\right] + \frac{1}{2}\frac{\partial}{\partial t}(\phi_0\hat{\mathrm{k}}\zeta_0),\end{aligned}\right\} \tag{5.1}$$

which is useful for the following reason. Each appearance of $\nabla^2, \hat{\mathrm{k}}$, or $\partial/\partial t$ in front of a quantity allows one to infer that the difference-frequency value of the corresponding Fourier factor is smaller than the sum-frequency value by the ratio $(\boldsymbol{k}-\boldsymbol{k}')^2/(\boldsymbol{k}+\boldsymbol{k}')^2$, $|\boldsymbol{k}-\boldsymbol{k}'|/|\boldsymbol{k}+\boldsymbol{k}'|$, or $(\omega-\omega')/(\omega+\omega')$. In addition, the phase quadrature (2.8) between ζ_0 and ϕ_0 implies the ratio $(\omega-\omega')/(\omega+\omega')$:

$$\begin{aligned}(\phi_0\zeta_0)^{(-)} &= \frac{1}{2}\,\mathrm{Re}\iint(\mathrm{i}a^*c)(a')\,\mathrm{e}^{\mathrm{i}(\theta'-\theta)}\,\mathrm{d}\boldsymbol{k}_c\,\mathrm{d}\boldsymbol{k}_c'\\ &= \frac{1}{4}\,\mathrm{Im}\iint a^*a'(c'-c')\,\mathrm{e}^{\mathrm{i}(\theta'-\theta)}\,\mathrm{d}\boldsymbol{k}_c\,\mathrm{d}\boldsymbol{k}_c',\end{aligned} \tag{5.2}$$

with the second step above coming from the symmetric interchange c.c. $+\,(\boldsymbol{k}\rightleftarrows\boldsymbol{k}')$. (A similar argument applies to ζ_0 and $\hat{\mathrm{k}}\phi_0$.) If we now limit the waves under consideration to a set that is narrow-band in both dimensions,

$$|\boldsymbol{k}-\boldsymbol{k}'|/|\boldsymbol{k}+\boldsymbol{k}'| < \beta, \tag{5.3}$$

with a small number β denoting half the relative bandwidth, we obtain

$$f^{(-)} = -\nabla\cdot\boldsymbol{m}^{(-)} + O(\beta^2)f^{(+)}, \qquad p^{(-)} = 0 + O(\beta^2)p^{(+)}, \tag{5.4}$$

so that to a quadratically good approximation in β we can neglect the pressure and retain only the leading flux term, which, though small to order β, can be recognized as the local rate of fluid accumulation from quadratic wave transport (2.4). As shown below it serves as the surface source of the balancing volume return flow.

When p is neglected in (2.6) one can solve for the perturbation potential as

$$\left(\frac{\partial^2}{\partial t^2} + g\hat{\mathrm{k}}\right)\phi_1^{(-)} = -gf^{(-)}, \tag{5.5}$$

which can be read as describing the response of low-wavenumber surface oscillators driven at frequencies well below their resonance by a fluctuating fluid source. The second time derivative can in fact be ignored to order β in this equation, as one can show by dividing by $g\hat{\mathrm{k}}$,

$$(1+\hat{\mathrm{F}}^{-1})\phi_1^{(-)} = -\hat{\mathrm{k}}^{-1}f^{(-)}, \tag{5.6}$$

with $\hat{\mathrm{F}}^{-1}$ standing for the inverse Froude-number operator,

$$\hat{\mathrm{F}}^{-1} = \frac{1}{g\hat{\mathrm{k}}}\frac{\partial^2}{\partial t^2}; \tag{5.7}$$

this quantity, the gravitational compliance of the surface at low difference frequencies, is small to order

$$\hat{\mathrm{F}}^{-1} \approx \frac{(\omega-\omega')^2}{g|\boldsymbol{k}-\boldsymbol{k}'|} = \frac{k-k'}{|\boldsymbol{k}-\boldsymbol{k}'|}\frac{\omega-\omega'}{\omega+\omega'} = O(\beta). \tag{5.8}$$

Equation (5.6) then represents flow under a flat surface – see (5.10) below – with the potential given by the right-hand side. Note that the definition of the operator $\hat{\mathrm{k}}$ implies

$$\phi_1^{(-)} \cong -\hat{\mathrm{k}}^{-1} f^{(-)}(\boldsymbol{x}) = -\int \frac{f^{(-)}(\boldsymbol{x}')}{2\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}', \tag{5.9}$$

which can be recognized as the surface value of a volume potential arising from the source density $f^{(-)} \cong -\nabla\cdot\boldsymbol{m}^{(-)}$, so that its surface gradient is the horizontal component of the return flow. The accompanying surface elevation can be evaluated similarly as

$$\zeta_1^{(-)} \cong -\frac{1}{g\hat{\mathrm{k}}}\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{m} = O(\beta)\zeta_1^{(+)}. \tag{5.10}$$

According to equation (2.3) the total horizontal second-order current at the surface is

$$\boldsymbol{u}_1^{(-)} = \nabla\phi_1^{(-)} + \boldsymbol{u}_s, \tag{5.11}$$

in which the second term, referred to here as the *Stokes current* by analogy to the classical drift, arises purely kinematically from the linear solution in the Hamiltonian formulation:

$$\boldsymbol{u}_s = -(w_0\nabla\zeta_0)^{(-)}. \tag{5.12}$$

It is easy to show from the definition $\boldsymbol{m} = -\phi\nabla\zeta$ that the return flow is smaller than the Stokes current by the factor β, so that $\boldsymbol{u}_s$ is in fact the dominant second-order low-frequency surface current for narrow-band wave sets. Note that in general both the wave momentum and the Stokes current have non-zero mean.

6. Crest sharpening

The non-dimensional surface strain field $\hat{\mathrm{k}}\zeta_0$ is numerically equivalent to the wave slope, the traditional measure of nonlinearity, but is in phase with the height, and for narrow-band wave groups very similar in profile. The quantity $\zeta_0\hat{\mathrm{k}}\zeta_0$ is the random analogue of the squared monochrome sinusoid, positive both for local maxima and minima of ζ_0 but with a positive average. The perturbation solution derived above, shorn of the local average,

$$\zeta_1^{(+)} = \zeta_0\hat{\mathrm{k}}\zeta_0 - (\zeta_0\hat{\mathrm{k}}\zeta_0)^{(-)} \tag{6.1}$$

is the analogue of the double-frequency sinusoid. Added to the linear solution it elevates both the crests and troughs, while depressing the intermediate heights.

However $\hat{\mathrm{k}}\zeta_0$ is not a perfect copy of ζ_0, because it contains the constituent Fourier components in different proportions. For a statistically homogeneous random height field the correlation ρ between the two,

$$\rho^2 = \frac{\langle\zeta_0\hat{\mathrm{k}}\zeta_0\rangle^2}{\langle\zeta_0^2\rangle\langle(\hat{\mathrm{k}}\zeta_0)^2\rangle}, \tag{6.2}$$

can be evaluated for a given wave-height spectrum $S(\boldsymbol{k})$ as

$$\rho^2 = \left(\int kS\,\mathrm{d}\boldsymbol{k}_c\right)^2\left(\int S\,\mathrm{d}\boldsymbol{k}_c\right)^{-1}\left(\int k^2S\,\mathrm{d}\boldsymbol{k}_c\right)^{-1} \tag{6.3}$$

and for a Phillips spectrum $S = Bk^{-4}$ band-limited between k_1 and $k_2 = k_1(1+2\beta)$, with otherwise arbitrary directional dependence, this becomes

$$\rho^2 = \frac{2\beta}{(1+\beta)\ln(1+2\beta)} \cong 1 - \beta^2/3. \tag{6.4}$$

The use of β here to denote half the relative bandwidth is intended to be compatible with the previous definition (5.3). For modest bandwidth the correlation is substantial, approaching unity for vanishing bandwidth as it should. The total second-order solution for height,

$$\zeta = \mathrm{Re}\big(\psi + \tfrac{1}{2}\psi\hat{\mathrm{k}}\psi\big) + \zeta_1^{(-)}, \tag{6.5}$$

(see (4.1)) clearly reproduces the ordinary Stokes solution in the limit of zero bandwidth. As a proper generalization to large bandwidth it should also account for the known behaviour of Stokes solutions with slowly varying amplitude and phase. That it does so can be demonstrated by applying (6.5) to an explicit narrow-band construction

$$\psi(\boldsymbol{x}, t) = \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{x} - \omega_0 t)} A(\boldsymbol{x}, t) \tag{6.6}$$

in which a slowly varying complex envelope A modulates a carrier sinusoid at central wavenumber $\boldsymbol{k}_0$. From the Fourier definition of $\hat{\mathrm{k}}$ one can deduce

$$\begin{aligned}\hat{\mathrm{k}}\psi(\boldsymbol{x}, t) &= \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{x} - \omega_0 t)}\big[k_0^2 - 2\mathrm{i}\boldsymbol{k}_0\cdot\nabla - \nabla^2\big]^{1/2} A(\boldsymbol{x}, t)\\ &= \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{x} - \omega_0 t)}\big[k_0 - \mathrm{i}\boldsymbol{\kappa}_0\cdot\nabla + \cdots\big] A(\boldsymbol{x}, t), \quad \boldsymbol{\kappa}_0 = \boldsymbol{k}_0/k_0,\end{aligned} \tag{6.7}$$

which shows that the second-order correction includes a small (order β) contribution from the random gradient of the envelope, as derived previously by Tayfun (1986) and Trulsen & Dysthe (1996). Note that in these traditional constructions the choice of central wavenumber is somewhat arbitrary; the results must nevertheless be invariant to small increments in the choice. The more general expression (6.5) is free of this arbitrariness.

At larger bandwidth, when the random elevation becomes increasingly chaotic, the crest sharpening occurs on average, but with more random local variation. If we assume that the complex height ψ is Gaussian with zero mean and variance $2\langle\zeta_0^2\rangle$, then for a particular value $\psi = |\psi|\,\mathrm{e}^{\mathrm{i}\varphi}$ the corresponding strain field has the conditional expectation and variance

$$\langle\hat{\mathrm{k}}\psi\rangle_\psi = \rho\varepsilon(\psi/h), \qquad \mathrm{var}_\psi(\hat{\mathrm{k}}\psi) = 2|\psi/h|^2\varepsilon^2(1-\rho^2), \tag{6.8}$$

where h and ε are the r.m.s. linear height and strain,

$$h^2 = \langle\zeta_0^2\rangle, \qquad \varepsilon^2 = \langle(\hat{\mathrm{k}}\zeta_0)^2\rangle. \tag{6.9}$$

The total wave height (6.5) can then be described, for a given value ψ of the linear solution, by the conditional normal distribution

$$\begin{aligned}\zeta &= \zeta_0 + \langle\zeta_1\rangle_\psi \pm (\zeta_1)_{rms} + \zeta_1^{(-)}\\ &= \zeta_1^{(-)} + |\psi|[(\cos\varphi + \varepsilon\rho|\psi/2h|\cos 2\varphi) \pm \varepsilon|\psi/2h|\sqrt{1-\rho^2}].\end{aligned} \tag{6.10}$$

Note that this expression predicts second-order crest sharpening for a progressive random-wave set of arbitrary bandwidth. Here, the slowly varying term $\zeta_1^{(-)}$ serves merely as a local mean level. In the narrow-band limit the quantity φ has the usual meaning as the phase of the modulated sinusoid, for which $\varphi = 0, \pi$ locates the crests and troughs. For wave sets of wider bandwidth and more disordered profiles the amplitude $|\psi|$ can vary substantially over a typical wavelength and interfere with this simple interpretation. It remains true nevertheless that the second-order term elevates the surface in the regions $\varphi \approx 0, \pm\pi$ inside which the actual local crests and troughs occur, while depressing the surface near the zero-crossing loci $\varphi = \pm\pi/2$ defining the wave 'shoulders' separating the crests and troughs. These effects, which occur preferentially at the larger non-dimensional amplitudes $|\psi/h|$, combine to sharpen on average the intervening crests and flatten the troughs.

The remaining random variability at second order in (6.10), present in addition to the average behaviour and induced by the finite bandwidth, is approximately proportional to the bandwidth; in the example (6.4) above, the fraction is

$$\sqrt{1-\rho^2} \cong \beta/\sqrt{3}, \tag{6.11}$$

which for a wave spectrum spanning one octave of wavenumber ($\beta = 1/3$) would amount to 19 % r.m.s.

7. Implications for crest-height statistics

Measured crest heights in the open ocean are known to exceed the Gaussian prediction by 10 %–20 % for the higher, less frequent events. Dynamical simulations of broad-band wave sets at second order account pretty well for the excess, as shown by Forristall (2000). On the other hand, corresponding formal predictions, based on narrow-band modulated Stokes waves, noticeably underpredict the heights. Can the present generalized representation improve the predictions? Unfortunately a detailed derivation of wave-height statistics is outside the scope of this paper; however, some observations may be in order.

In the narrow-band limit the normalized crest amplitudes of the linear solution y are predicted by the envelope amplitude itself, which for a Gaussian process obeys the Rayleigh exceedance distribution

$$P_R(y) = \exp(-y^2/2); \tag{7.1}$$

the second-order Stokes representation of normalized elevation z,

$$z = y + \tfrac{1}{2}\varepsilon_0 y^2, \qquad \varepsilon_0 = k_0 h, \tag{7.2}$$

solved for $y(z)$, then yields the probability for the nonlinear crest height to exceed z (Tayfun 1980):

$$P_>(z) = P_R(y(z)), \qquad y(z) = \varepsilon_0^{-1}\left[(1+2\varepsilon_0 z)^{1/2} - 1\right]. \tag{7.3}$$

The present broad-band generalization to (7.2) is

$$z = y + \tfrac{1}{2}\varepsilon y^2\left[\rho + w(1-\rho^2)^{1/2}\right], \qquad y = |\psi|/h \tag{7.4}$$

(see (6.8)–(6.10)), under the simplifying assumption that one can neglect $\zeta_1^{(-)}$ and take $\varphi = 0$ as defining the crests. Here w is an independent Gaussian random variable of unit variance. Note that this expression reverts to (7.2) in the narrow-band limit, as $\rho \to 1$. For a broad-band Gaussian process y the exceedance formula (7.1) for maxima must be replaced by a more general formula given by Rice (1944, equation 3.6–11), because the irregular envelope amplitude is no longer a reliable guide to the distribution of maxima. As it happens, the exceedance probability in this case is asymptotically the same as $P_R(y)$ for the higher crests, when $y^2 \gg 1$. The implied nonlinear crest-height probability can then be computed as in (7.2) and (7.3), with the replacement

$$\varepsilon_0 \to \varepsilon\left[\rho + w(1-\rho^2)^{1/2}\right], \tag{7.5}$$

and the result averaged over w. It is not hard to show that this replacement in fact increases the exceedance frequency $P_>(z)$. Whether this increase makes up the defect in the existing formula remains to be seen.

8. Discussion

The simple quadratic expression (6.5) for elevation, along with its accompanying surface potential (2.8), (4.1), is a natural generalization of the second-order, slowly varying Stokes solution to random wave sets of finite bandwidth. It is an exact solution to the leading perturbation correction in one surface dimension and a good approximation for long-crested waves in two surface dimensions. Its factored form in spatial coordinates lends itself to a simple formal explanation of crest sharpening in disordered, naturally occurring wave sets. For purposes of numerical simulation it has the advantage of economy as well, requiring only one additional Fourier transform to compute the nonlinear correction to the elevation.

It should be noted that when the relative bandwidth is not small, the difference-beat elevation solution $\zeta_1^{(-)}$ is no longer negligible as in § 5. However, for $\beta < 1/3$ the difference wavenumbers are all less than the lower limit of the primary band and these terms cannot interfere with the second-order distortion of the waveforms worked out here. The present solution can therefore be relied on as a useful description of random wave sets spanning up to an octave of wavenumber.

On the other hand, as β approaches unity the ratio of maximum to minimum wavelength becomes large; the sum and difference beats generated by the longest waves then take the form of sideband pairs accompanying the shorter waves, a form which accounts for the advective displacement of the short waves by the long-wave orbital currents. The simple sum-beat solution developed here, while still valid, does not include these effects.

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